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The embedding $U'_q(so_3) \subset U_q(sl_3)$

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Abstract

We study the embedding $U'_q(so_3) \subset U_q(sl_3)$, where $U_q(sl_3)$ is a well known Drinfeld–Jimbo quantum algebra and the algebra $U'_q(so_3)$ is the cyclically symmetric q-deformation of the universal enveloping algebra $U(so_3)$ of the Lie algebra so_3 which is not a Drinfeld–Jimbo quantum algebra. Finite-dimensional irreducible representations of $U_q(sl_3)$ are decomposed into irreducible representations of $U'_q(so_3)$. An explicit expression for the matrix of the transition from the Gel'fand–Tsetlin basis for $U_q(sl_3)$ to the bases of irreducible representations of $U'_q(so_3)$ is calculated for representations of $U_q(sl_3)$ with highest weights $(l, 0, 0)$. Entries of this matrix are expressed in terms of products of dual q-Krawtchouk polynomials and dual q-Hahn polynomials. Expressions for representation operators of $U_q(sl_3)$ in the $U'_q(so_3)$ basis are given.

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1. Introduction

The embedding $SO(3) \subset SU(3)$ is of a great importance for physics [1–3]. On the Lie algebra level, this embedding is fulfilled by choosing the elements $E_{12} - E_{21}$, $E_{13} - E_{31}$, $E_{23} - E_{32}$ of the Lie algebra su$_3$ as a basis for the Lie subalgebra so_3, where E_{ij} are the matrices with entries $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$. The embedding $SO(3) \subset SU(3)$ differs essentially from the embedding $SU(2) \subset SU(3)$.

After the appearance of quantum groups and quantum algebras, much attention was paid to the construction of q-analogues of the embedding $SO(3) \subset SU(3)$ (see, for example, [4–6] and references therein). It is clear now that it is not possible to construct an embedding $U_q(so_3) \subset U_q(sl_3)$ in such a way that $U_q(sl_3)$ and $U_q(so_3)$ would be a Hopf algebra and its Hopf subalgebra, respectively (see [7], chapter 1, for the corresponding definitions). In constructing an embedding $U_q(so_3) \subset U_q(sl_3)$ one usually tries to take the standard Drinfeld–Jimbo algebra $U_q(sl_2)$ (or an algebra isomorphic to it) as the subalgebra $U_q(so_3)$.

Our idea in this paper is the following. If we wish to construct a q-deformation of the embedding $so_3 \subset sl_3$, then we have to make a q-deformation of the commutation relations for the basis $I'_{21} \equiv E_{12} - E_{21}$, $I'_{31} \equiv E_{13} - E_{31}$, $I'_{32} \equiv E_{23} - E_{32}$ of the subalgebra so_3. It is possible to construct such a q-deformation. The role of $U_q(so_3)$ in this q-deformed embedding

$U_q(so_3) \subset U_q(sl_3)$ is the cyclically symmetric algebra generated by the elements I_{21}, I_{31}, I_{32}, satisfying the relations

$$[I_{21}, I_{32}]_q = I_{31} \qquad [I_{32}, I_{31}]_q = I_{21} \qquad [I_{31}, I_{21}]_q = I_{32}$$

where the q-commutator $[A, B]_q$ is defined by $[A, B]_q = q^{1/2}AB - q^{-1/2}BA$. This algebra (we denote it by $U'_q(so_3)$) was introduced by Fairlie [8]. It is isomorphic to the algebra studied by Odesski [9]. It was shown in [10] that $U'_q(so_3)$ is not isomorphic to the quantum algebra $U_q(sl_2)$ and can be embedded into a certain extension of $U_q(sl_2)$.

The elements I_{21}, I_{31}, I_{32} of $U'_q(so_3)$ are q-deformations of the elements $E_{12} - E_{21}$, $E_{13} - E_{31}$, $E_{23} - E_{32}$ of the subalgebra so_3 of sl_3 and at $q \to 1$ they turn into these elements of so_3. This means that $U'_q(so_3) \subset U_q(sl_3)$ is a natural q-deformation of the embedding $so_3 \subset sl_3$ and may be important for construction of q-deformations of the 'classical' models in nuclear physics. Some other motivations for studying the embedding $U'_q(so_3) \subset U_q(sl_3)$ are the following.

(a) The algebra $U'_q(so_3)$ arises naturally as the algebra of observables in 2 + 1 quantum gravity on the torus (see, for example, [11]). This algebra appears in [12] in the study of geometry on Teichmüller spaces. It also appears in the theory of links in the algebraic topology [13]. The embedding $U'_q(so_3) \subset U_q(sl_3)$ allows us to use the algebra $U_q(sl_3)$ for studying properties of $U'_q(so_3)$. For example, using this embedding it is shown in [14] that $U'_q(so_3)$ has no divisors of zero and that finite-dimensional representations of the algebra $U'_q(so_3)$ separate its elements.

(b) The embedding $U'_q(so_3) \subset U_q(sl_3)$ is also used for the construction of a quantum analogue of the symmetric Riemannian space $SU(3)/SO(3)$ (see [15]). For studying harmonic analysis on this quantum symmetric space we need to know the relationship between finite-dimensional representations of $U'_q(so_3)$ and $U_q(sl_3)$. We study this relationship in the present paper.

(c) A q-analogue of the theory of harmonic polynomials (q-harmonic polynomials) on the three-dimensional quantum vector space $\mathbb{R}^3_q$ is constructed by using the algebra $U_q(sl_3)$ and its subalgebra $U'_q(so_3)$. (This theory is contained in [16, 17].) If we wish to know how elements of the algebra $U_q(sl_3)$ act upon basis q-harmonic polynomials, we have to use the results of the present paper.

(d) The concrete results of this paper on the embedding $U'_q(so_3) \subset U_q(sl_3)$ can be used for studying q-orthogonal polynomials. Namely, it is known (see [15]) that zonal spherical functions on the q-analogue of the quantum space $SU(3)/SO(3)$ are expressed in terms of symmetric Macdonald polynomials. The results of our paper allows us to connect symmetric Macdonald polynomials with dual q-Krawtchouk polynomials and dual q-Hahn polynomials. The results on this subject will be published in a separate paper.

A shortcoming of the algebra $U'_q(so_3)$ is that a Hopf algebra structure is not known on it. Nevertheless, it is a coideal of $U_q(sl_3)$. Moreover, it is possible to construct tensor products of finite-dimensional irreducible representations of $U'_q(so_3)$ (usually, a Hopf algebra structure is used for such a construction for Drinfeld–Jimbo quantum algebras). The details of such a construction see [10].

The aim of this paper is to study the embedding $U'_q(so_3) \subset U_q(sl_3)$ with the subalgebra $U'_q(so_3)$ defined above. First, we restrict finite-dimensional irreducible representations of $U_q(sl_3)$ to $U'_q(so_3)$ and decompose these restrictions into irreducible representations of $U'_q(so_3)$. For this decomposition we diagonalize (by using the dual q-Krawtchouk polynomials) the operators $T(I_{21})$ of irreducible representations T of $U_q(sl_3)$. We show that this decomposition is the same as in the classical case $SO(3) \subset SU(3)$. Then we restrict ourselves only by irreducible representations $T_{(l,0,0)}$ of $U_q(sl_3)$ with highest weights $(l,0,0)$ and construct explicitly $U'_q(so_3)$ bases in the spaces of these representations. In order to construct them we use properties of dual q-Hahn polynomials. Products of dual q-Krawtchouk polynomials and dual q-Hahn polynomials constitute coefficients of transition from the Gel'fand–Tsetlin basis for the irreducible representation $T_{(l,0,0)}$ to the $U'_q(so_3)$ basis. Then we derive how operators of the representation $T_{(l,0,0)}$ of $U_q(sl_3)$ act upon the $U'_q(so_3)$ basis elements. In fact, we derive a formula of action of $T_{(l,0,0)}$ upon the last basis only for the single operator $T_{(l,0,0)}(q^{2H_3})$, for which the action formula is simplest. (Here q^{2H_3} is the element of the Cartan subalgebra defined at the end of section 2.) As in the classical case, the corresponding formulae for other operators $T_{(l,0,0)}(a)$ with generating elements a of $U_q(sl_3)$ can be found by commuting (or q-commuting) q^{2H_3} with elements of $U'_q(so_3)$. The action formulae for the representation operators corresponding to the last elements are known.

2. The quantum algebra $U_q(sl_3)$ and its subalgebra $U'_q(so_3)$

The quantum algebra $U_q(sl_3)$ is generated by the elements e_1, f_1, e_2, f_2, $k_1 = q^{h_1}$, $k_2 = q^{h_2}$ satisfying the relations

$$k_1k_2 = k_2k_1 \qquad k_ik_i^{-1} = k_i^{-1}k_i = 1 \qquad i = 1,2$$

$$k_ie_jk_i^{-1} = q^{a_{ij}}e_j \qquad k_if_jk_i^{-1} = q^{-a_{ij}}f_j \qquad [e_i, f_j] = \delta_{ij}\frac{k_i - k_i^{-1}}{q - q^{-1}}$$

$$e_i^2e_{i\pm1} - (q+q^{-1})e_ie_{i\pm1}e_i + e_{i\pm1}e_i^2 = 0$$

$$f_i^2f_{i\pm1} - (q+q^{-1})f_if_{i\pm1}f_i + f_{i\pm1}f_i^2 = 0$$

where $a_{ii} = 2$ and $a_{12} = a_{21} = -1$ (see [18, 19]).

Let us take in $U_q(sl_3)$ the elements

$$I_{21} = f_1 - qq^{-h_1}e_1 \qquad I_{32} = f_2 - qq^{-h_2}e_2.$$

Direct calculation shows that these elements satisfy the relations

$$I_{21}^2I_{32} - (q+q^{-1})I_{21}I_{32}I_{21} + I_{32}I_{21}^2 = -I_{32} \tag{1}$$

$$I_{21}I_{32}^2 - (q+q^{-1})I_{32}I_{21}I_{32} + I_{32}^2I_{21} = -I_{21}. \tag{2}$$

Introducing the element $I_{31} = q^{1/2}I_{21}I_{32} - q^{-1/2}I_{32}I_{21}$, we obtain the quadratic relations for I_{21}, I_{32} and I_{31}:

$$q^{1/2}I_{21}I_{32} - q^{-1/2}I_{32}I_{21} = I_{31} \tag{3}$$

$$q^{1/2}I_{31}I_{21} - q^{-1/2}I_{21}I_{31} = I_{32} \tag{4}$$

$$q^{1/2}I_{32}I_{31} - q^{-1/2}I_{31}I_{32} = I_{21} \tag{5}$$

(the relations (4) and (5) are consequences of (1)–(3)). The associative algebra (with a unity) generated by the elements I_{21} and I_{32} satisfying the relations (1) and (2) or, equivalently, the associative algebra (with unit element) generated by I_{21}, I_{32} and I_{31} satisfying the relations (3)–(5) is denoted by $U'_q(so_3)$. It is a q-deformation of the universal enveloping algebra $U(so_3)$ of the Lie algebra so_3 (for $q = 1$ the relations (3)–(5) turn into the well known commutation relations for the basis elements of so_3).

Note that the relations (1) and (2) are a special case of the relations defining the non-standard q-deformation $U'_q(so_n)$ of the universal enveloping algebra of the Lie algebra so_n (see [20]). The relations (3)–(5) appeared in the paper by Fairlie [8].

Sometimes, it is useful to extend the embedding $U'_q(so_3) \subset U_q(sl_3)$ to the embedding $U'_q(so_3) \subset U_q(gl_3)$. Instead of elements $k_1 = q^{h_1}$ and $k_2 = q^{h_2}$, in $U_q(gl_3)$ we have the elements q^{H_1}, q^{H_2} and q^{H_3} such that $q^{h_1} = q^{H_2}q^{-H_1}$ and $q^{h_2} = q^{H_3}q^{-H_2}$ (for a complete definition of $U_q(gl_3)$ see, for example, [7], chapter 6).

Throughout the rest of the paper we suppose that q is a real number such that $0 < q < 1$.

3. Finite-dimensional representations of $U'_q(so_3)$

The algebra $U'_q(so_3)$ has two types of irreducible finite-dimensional representations: representations of the classical type and representations of the non-classical type (see [10]). As in the case of the irreducible representations of the Lie algebra so_3, irreducible representations of the classical type are given by a non-negative integral or half-integral number r and are denoted by T'_r. The representation T'_r acts on a $(2r+1)$-dimensional vector space V_r. There exists a basis $|r, x\rangle$, $x = -r, -r+1, \ldots, r$, in V_r such that the representation operators $T'_r(I_{21})$ and $T'_r(I_{32})$ are given by the formulae

$$T'_r(I_{21})|r, x\rangle = \mathrm{i}[x]|r, x\rangle \tag{6}$$

$$T'_r(I_{32})|r, x\rangle = \frac{[x]}{[2x]}\{([r-x][r+x+1])^{1/2}|r, x+1\rangle - ([r-x+1][r+x])^{1/2}|r, x-1\rangle\} \tag{7}$$

where $[a] \equiv [a]_q$ means a q-number defined as

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}.$$

Irreducible representations of non-classical type are given by a non-negative integer n and by numbers ϵ_1, ϵ_2 taking the values 0 or 1. The corresponding representations are denoted by $R_{n\epsilon_1\epsilon_2}$ and act on an n-dimensional vector space V_n with the basis $|1\rangle, |2\rangle, \ldots, |n\rangle$. The operators $R_{n\epsilon_1\epsilon_2}(I_{21})$ and $R_{n\epsilon_1\epsilon_2}(I_{32})$ are given by the formulae

$$\begin{aligned} R_{n\epsilon_1\epsilon_2}(I_{21})|k\rangle &= \epsilon_1 \frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}}|k\rangle \\ R_{n\epsilon_1\epsilon_2}(I_{32})|1\rangle &= \epsilon_2 \frac{[n]}{q^{1/2} - q^{-1/2}}|1\rangle + \mathrm{i}\frac{[n-1]}{q^{1/2} - q^{-1/2}}|2\rangle \\ R_{n\epsilon_1\epsilon_2}(I_{32})|k\rangle &= \mathrm{i}\frac{q^k[n-k]}{q^{k-1/2} - q^{-k+1/2}}|k+1\rangle + \mathrm{i}\frac{q^{-k+1}[n+k-1]}{q^{k-1/2} - q^{-k+1/2}}|k-1\rangle \end{aligned} \tag{8}$$

where $k \neq 1$ in the last formula and $\mathrm{i} = \sqrt{-1}$.

It follows from (6) and (8) that the spectra of the operators $T'_r(I_{21})$ and $R_{n\epsilon_1\epsilon_2}(I_{21})$ consist of the points

$$\mathrm{i}[x] \qquad x = -r, -r+1, \ldots, r \tag{9}$$

and of the points

$$\epsilon_1 \frac{q^{k-1/2} + q^{-k+1/2}}{q - q^{-1}} \qquad k = 1, 2, \ldots, n \tag{10}$$

respectively. Thus, for positive q the spectrum of the representation operator $T(I_{21})$ consists of pure imaginary points for irreducible representations of the classical type and of real points for irreducible representations of the non-classical type.

We need below the following assertion: *every finite-dimensional irreducible representation of $U'_q(so_3)$ is equivalent to one of the representations T'_r of classical type or to one of the representations $R_{n\epsilon_1\epsilon_2}$ of non-classical type.* This assertion is given in [9] for the algebra isomorphic to $U'_q(so_3)$.

Taking into account this assertion, we derive from formulae (9) and (10) that *a finite-dimensional irreducible representation T of $U'_q(so_3)$ is uniquely (up to an equivalence) determined by the spectrum of the operator $T(I_{21})$.*

4. Representations of the quantum algebra $U_q(sl_3)$

We consider irreducible finite-dimensional representations T_ℓ of the quantum algebra $U_q(sl_3)$ given by integers $\ell \equiv (l_1, l_2, l_3)$ such that $l_1 \geqslant l_2 \geqslant l_3$. The representation T_ℓ acts on the vector space V_ℓ with the Gel'fand–Tsetlin basis

$$|m_1, m_2; m\rangle \qquad l_1 \geqslant m_1 \geqslant l_2 \geqslant m_2 \geqslant l_3 \qquad m_1 \geqslant m \geqslant m_2.$$

The operators of this representation act on these basis vectors as

$$T_\ell(q^{h_1})|m_1, m_2; m\rangle = q^{2m-m_1-m_2}|m_1, m_2; m\rangle \tag{11}$$

$$T_\ell(q^{h_2})|m_1, m_2; m\rangle = q^{2m_1+2m_2-l_1-l_2-l_3-m}|m_1, m_2; m\rangle \tag{12}$$

$$T_\ell(e_1)|m_1, m_2; m\rangle = ([m_1 - m][m - m_2 + 1])^{1/2}|m_1, m_2; m + 1\rangle$$

$$T_\ell(f_1)|m_1, m_2; m\rangle = ([m_1 - m + 1][m - m_2])^{1/2}|m_1, m_2; m - 1\rangle$$

(the operators $T_\ell(e_2)$ and $T_\ell(f_2)$ will be given below). In the next section we shall need the basis $|m_1, m_2; m\rangle'$ of the space V_ℓ given by

$$|m_1, m_2; m\rangle' = q^{(m_1-m)(m-m_2)/2}|m_1, m_2; m\rangle.$$

The operators $T_\ell(q^{h_1})$ and $T_\ell(q^{h_2})$ act upon this new basis by the same formulae as upon the previous basis, that is by the formulae (11) and (12). For $T_\ell(e_1)$ and $T_\ell(f_1)$ we have

$$T_\ell(e_1)|m_1, m_2; m\rangle' = q^{(2m-m_1-m_2+1)/2}([m_1 - m][m - m_2 + 1])^{1/2}|m_1, m_2; m + 1\rangle'$$

$$T_\ell(f_1)|m_1, m_2; m\rangle' = q^{(-2m+m_1+m_2+1)/2}([m_1 - m + 1][m - m_2])^{1/2}|m_1, m_2; m - 1\rangle'.$$

The representation T_ℓ can be extended to the representation of the quantum algebra $U_q(gl_3)$ (we denote this extended representation by the same symbol T_ℓ). For this extension we have

$$T_\ell(q^{H_1})|m_1, m_2; m\rangle' = q^{-m}|m_1, m_2; m\rangle'$$

$$T_\ell(q^{H_2})|m_1, m_2; m\rangle' = q^{m-m_1-m_2}|m_1, m_2; m\rangle'$$

$$T_\ell(q^{H_3})|m_1, m_2; m\rangle' = q^{m_1+m_2-l_1-l_2-l_3}|m_1, m_2; m\rangle'.$$

We shall need these formulae in section 8.

5. Restriction of the representations of $U_q(sl_3)$ to $U'_q(so_3)$

The aim of this section is to restrict the irreducible representation T_ℓ, $\ell = (l_1, l_2, l_3)$, of $U_q(sl_3)$ to the subalgebra $U'_q(so_3)$ and to decompose this restriction into irreducible representations of $U'_q(so_3)$. In order to do this we consider the representation T_ℓ for elements of $U'_q(so_3)$. We have

$$\begin{aligned} T_\ell(I_{21})|m_1, m_2; m\rangle' &= (T_\ell(f_1) - qT_\ell(q^{-h_1})T_\ell(e_1))|m_1, m_2; m\rangle' \\ &= q^{(m_1+m_2-2m+1)/2}([m_1 - m + 1][m - m_2])^{1/2}|m_1, m_2; m-1\rangle' \\ &\quad -q^{(m_1+m_2-2m-1)/2}([m_1 - m][m - m_2 + 1])^{1/2}|m_1, m_2; m+1\rangle'. \end{aligned} \tag{13}$$

Let us diagonalize this operator. This means that we have to find in the space V_ℓ the vectors

$$|m_1, m_2; x\rangle' = \sum_{m=m_2}^{m_1} a_m^x |m_1, m_2; m\rangle' \tag{14}$$

such that

$$T_\ell(I_{21})|m_1, m_2; x\rangle' = \mathrm{i}[x]|m_1, m_2; x\rangle' \tag{15}$$

where $\mathrm{i} = \sqrt{-1}$ and $[x]$ is a q-number. We have to find eigenvalues $\mathrm{i}[x]$. (Note that representing these eigenvalues in the form $\mathrm{i}[x]$ we do not restrict ourselves since for a fixed positive q the number x may take any complex number.) In order to find eigenvalues in (15) we act by the operator $T_\ell(I_{21})$ upon both sides of (14):

$$\mathrm{i}[x]|m_1, m_2; x\rangle' = \sum_{m=m_2}^{m_1} a_m^x T_\ell(I_{21})|m_1, m_2; m\rangle'.$$

We substitute into the right-hand side of this relation the expression (13) for the vectors $T_\ell(I_{21})|m_1, m_2; m\rangle'$ and into the left-hand side the expression (14) for the vector $|m_1, m_2; x\rangle'$. Then we compare coefficients at the vector $|m_1, m_2; m\rangle'$. As a result, we obtain a recurrence relation for computation of the coefficients a_m^x:

$$a_{m+1}^x q^{-1/2} d\sqrt{[m_1 - m][m - m_2 + 1]} - a_{m-1}^x q^{1/2} d\sqrt{[m_1 - m + 1][m - m_2])} = \mathrm{i}[x]a_m^x \tag{16}$$

where $d = q^{(m_1+m_2-2m)/2}$. Using the notation

$$N = m_1 - m_2 \qquad n = m - m_2 \qquad P_n(x) = a_{n+m_2}^x$$

we obtain

$$q^{(N-2n-1)/2}\{\sqrt{[N-n][n+1]}P_{n+1}(x) - q\sqrt{[N-n+1][n]}P_{n-1}(x)\} = \mathrm{i}[x]P_n(x).$$

Making here the substitution

$$P_n(x) = \mathrm{i}^{-n}\left(\frac{[N]!}{[n]![N-n]!}\right)^{1/2} q^{-n(N-1)/2} P'_n(x)$$

where $[n]! = [n][n-1][n-2]\cdots[1]$, we obtain the recurrence relation

$$(1 - q'^{n-N})P'_{n+1}(x) - q'^{-N}(1 - q'^n)P'_{n-1}(x) = q'^{-N/2}(q'^{-x/2} - q'^{x/2})P'_n(x)$$

where $q' = q^{-2}$. We compare this relation with the recurrence relation (see [21])

$$\begin{aligned} c\hat{q}(\hat{q}^n - 1)k_{n-1}(\lambda(y)) + (\hat{q}^{-N} - c\hat{q})\hat{q}^n k_n(\lambda(y)) + (1 - \hat{q}^{n-N})k_{n+1}(\lambda(y)) \\ = (\hat{q}^{-y} - c\hat{q}^{y+1})k_n(\lambda(y)) \end{aligned}$$

for the dual q-Krawtchouk polynomials

$$k_n(\lambda(y); c, N \mid \hat{q}) = {}_3\varphi_2(\hat{q}^{-y}, -c\hat{q}^{y+1}, \hat{q}^{-n};\ 0, \hat{q}^{-N};\ \hat{q}, \hat{q})$$

with $\lambda(y) = \hat{q}^{-y} - c\hat{q}^{y+1}$, where ${}_3\varphi_2$ is a basic hypergeometric function (see [22] or [23], section 13.2.2). Then at $\hat{q} = q' = q^{-2}$, $c = q'^{-N-1} = q^{2N+2}$ and $x = 2y - N$ we obtain

$$P_n(x) = \mathrm{i}^{-n} \left(\frac{[N]!}{[n]![N-n]!} \right)^{1/2} q^{-n(N-1)/2} k_n(\lambda(y));\ q^{2N+2}, N \mid q^{-2})$$

where $\lambda(y)) = q^{x+N} - q^{-x+N}$. Therefore,

$$a_m^x = P_{m-m_2}(x) = \mathrm{i}^{m_2-m} q^{-(m-m_2)(m_1-m_2-1)/2} \left(\frac{[m_1-m_2]!}{[m-m_2]![m_1-m]!} \right)^{1/2}$$
$$\times\, {}_3\varphi_2(q^{x+m_1-m_2}, -q^{x+m_2-m_1}, q^{2(m-m_2)};\ 0, q^{2(m_1-m_2)};\ q^{-2}, q^{-2})$$

where m runs over the values $m_2, m_2+1, \ldots, m_1$, that is, $n = m - m_2$ runs over $0, 1, 2, \ldots, N = m_1 - m_2$.

The dual q-Krawtchouk polynomials are orthogonal on the set $y = 0, 1, 2, \ldots, N$ (see [21]). This means that the rows and columns of the $(N+1) \times (N+1)$ matrix a with entries a_m^x are orthogonal to each other, and we obtain the vectors $|m_1, m_2;\ x\rangle'$ for the following values of $x = 2y - N$:

$$x \in \{-N, -N+2, \ldots, N\} \equiv \{-(m_1-m_2), -(m_1-m_2)+2, \ldots, m_1-m_2\}$$

(since x runs over these values when y runs over the values $0, 1, 2, \ldots, N$). Therefore, for each fixed m_1 and m_2 we obtain the spectral points

$$-\mathrm{i}[m_1-m_2],\ -\mathrm{i}[m_1-m_2-2],\ \ldots,\ \mathrm{i}[m_1-m_2] \tag{17}$$

of the operator $T_\ell(I_{21})$. The whole spectrum of this operator in the representation T_ℓ consists of the family of sets (17) (we denote these sets by $S(m_1, m_2)$) taken for all integral values m_1 and m_2 such that $l_1 \geqslant m_1 \geqslant l_2 \geqslant m_2 \geqslant l_3$, that is,

$$\mathrm{Spec}\ T_\ell(I_{21}) = \bigcup_{l_1 \geqslant m_1 \geqslant l_2 \geqslant m_2 \geqslant l_3} S(m_1, m_2).$$

Thus, the spectrum of $T_\ell(I_{21})$ consists only of purely imaginary points. Comparing this spectrum with the spectra of operators $T(I_{21})$ of irreducible representations T of $U'_q(so_3)$ given in section 3 we conclude that *the restriction $T_\ell{\downarrow}_{U'_q(so_3)}$ decomposes only into irreducible representations of the classical type* (since only they give purely imaginary points in the spectra).

In order to find which irreducible representations of $U'_q(so_3)$ are contained in the decomposition of $T_\ell{\downarrow}_{U'_q(so_3)}$ we have to split the spectrum Spec $T_\ell(I_{21})$ into spectra of operators $T(I_{21})$ of irreducible representations T of $U'_q(so_3)$. In order to do this we note that the set $S(m_1, m_2)$ is not a spectrum of some irreducible representation of $U'_q(so_3)$. We have

$$S(m_1, m_2) \bigcup S(m_1-1, m_2) = \mathrm{Spec}\ T'_{m_1-m_2}(I_{21}) \tag{18}$$

where $T'_{m_1-m_2}$ is the irreducible representation of $U'_q(so_3)$ from section 3. Splitting Spec $T_\ell(I_{21})$ into parts of type (18) we conclude that

$$T_\ell{\downarrow}_{U'_q(so_3)} = {\sum_s}' \sum_{k=s}^{s+l_2-l_3} T'_k \tag{19}$$

if $l_1 - l_2$ is odd and

$$T_\ell{\downarrow}_{U'_q(so_3)} = {\sum_s}' \sum_{k=s}^{s+l_2-l_3} T'_k \oplus {\sum_r}' T'_r \tag{20}$$

if $l_1 - l_2$ is even, where $\sum'_s$ denotes summation over the values $l_1 - l_2, l_1 - l_2 - 2, l_1 - l_2 - 4, \ldots, 1$ (or 2) and the last sum $\sum'_r$ is over the values $l_2-l_3, l_2-l_3-2, l_2-l_3-4, \ldots, 0$ (or 1).

The decompositions (19) and (20) are direct sum decompositions since $T_\ell \downarrow_{U'_q(so_3)}$ is a completely reducible representation (see [16]). Note that the decompositions (19) and (20) coincide with the corresponding decompositions for the reduction $SU(3) \supset SO(3)$ (see, for example, [24]).

The orthogonality relation for the coefficients a^x_m from (14) follows from the orthogonality for the dual q-Krawtchouk polynomials which can be found in [21]. Using the last orthogonality we find that

$$\sum_{x=-(m_1-m_2)}^{m_1-m_2} a^x_m \overline{a^x_n} W(x) = \delta_{mn} \tag{21}$$

where the summation is with step 2 and

$$W(x) = q^{(x^2-m_1+m_2)/2} \frac{[2x][m_1-m_2]!}{2[x][m_1-m_2+x]!![m_1-m_2-x]!!} \tag{22}$$

with $[m]!! = [m][m-2][m-4]\cdots[0]$ (or [1]).

Formula (21) shows that the vectors (14) are not normalized. The vectors

$$|m_1, m_2\ x\rangle'' = W(x)^{1/2} |m_1, m_2\ x\rangle' \tag{23}$$

are normal and $T_\ell(I_{2,1})|m_1, m_2\ x\rangle'' = \mathrm{i}[x]|m_1, m_2\ x\rangle''$.

6. The action formula for the operator $T_\ell(I_{32})$

Below we consider the irreducible representations of $U_q(sl_3)$ only with highest weights $(l, 0, 0)$. We denote these representations by $T_{l,\mathbf{0}}$. We derive from (19) and (20) that

$$T_{l,\mathbf{0}}{\downarrow}_{U'_q(so_3)} = T'_l \oplus T'_{l-2} \oplus T'_{l-4} \oplus \cdots \oplus T'_0 \text{ (or } T'_1).$$

Let us find how the operator $T_{l,\mathbf{0}}(I_{32})$ acts upon the basis elements $|m_1, 0;\ x\rangle'' \equiv |m_1;\ x\rangle''$ given by (23). Since

$$T_{l,\mathbf{0}}(e_2)|m_1;\ m\rangle' = q^{-m/2}([l-m_1][m_1-m+1])^{1/2}|m_1+1;\ m\rangle'$$

$$T_{l,\mathbf{0}}(f_2)|m_1;\ m\rangle' = q^{m/2}([l-m_1+1][m_1-m])^{1/2}|m_1-1;\ m\rangle'$$

$$T_{l,\mathbf{0}}(q^{h_2})|m_1;\ m\rangle' = q^{2m_1-l-m}|m_1;\ m\rangle'$$

where the vectors $|m_1, 0;\ m\rangle'$ are denoted by $|m_1;\ m\rangle'$, then

$$T_{l,\mathbf{0}}(I_{32})|m_1;\ m\rangle' = q^{m/2}\{([l-m_1+1][m_1-m])^{1/2}|m_1-1;\ m\rangle' \\ -q^{l-2m_1-1}([l-m_1][m_1-m+1])^{1/2}|m_1+1;\ m\rangle'\}.$$

Therefore, using relation (14) we have

$$T_{l,\mathbf{0}}(I_{32})|m_1;\ x\rangle'' = \sum_{m=0}^{m_1} c^x_m(m_1) q^{m/2}\sqrt{[l-m_1+1][m_1-m]}|m_1-1;\ m\rangle' \\ - \sum_{m=0}^{m_1} c^x_m(m_1) q^{l-2m_1-1+m/2}\sqrt{[l-m_1][m_1-m+1]}|m_1+1;\ m\rangle' \tag{24}$$

where $|m_1;\ x\rangle'' \equiv |m_1, 0;\ x\rangle''$, $c^x_m(m_1) = W(x)^{-1/2} a^x_m$ and $W(x)$ is given by formula (22) at $m_2 = 0$. We have to express the sums on the right-hand side as linear combinations of vectors of the type $|m_1;\ x\rangle''$. (Note that it follows from the results of section 5 that m_1 and x are of the same evenness.) For this we need the relation

$$(\alpha - a)(1-f)\ {}_3\varphi_2(\alpha, a, b;\ d, f;\ \hat{q}, z) = (\alpha - f)(1-a)\ {}_3\varphi_2(\alpha, a\hat{q}, b;\ d, f\hat{q};\ \hat{q}, z)$$
$$+(1-\alpha)(f-a)\ {}_3\varphi_2(\alpha\hat{q}, a, b;\ d, f\hat{q};\ \hat{q}, z) \qquad (25)$$

which is proved by a direct calculation by comparing coefficients at z^n. Using this formula at

$$\hat{q} = q^{-2} \qquad b = q^{2m} \qquad d = 0 \qquad f = q^{2m_1}$$
$$\alpha = q^{x+m_1} \qquad a = -q^{m_1 - x} \qquad z = q^{-2}$$

and the expression for $c^x_m(m_1)$ in terms of the q-hypergeometric function ${}_3\varphi_2$ we obtain the following recurrence relation for the coefficients $c^x_m(m_1)$:

$$c^x_m(m_1) = \left(\frac{[x]}{[2x][m_1 - m]}\right)^{1/2} \left\{ q^{(x-m-1)/2} \left(\frac{[m_1 + x][x-1]}{[2x-2]}\right)^{1/2} c^{x-1}_m(m_1 - 1)\right.$$
$$\left. + q^{-(x+m+1)/2} \left(\frac{[m_1 - x][x+1]}{[2x+2]}\right)^{1/2} c^{x+1}_m(m_1 - 1)\right\}. \qquad (26)$$

Next we need the relation

$${}_3\varphi_2(\hat{q}^{-n}, a, b;\ d, f;\ \hat{q}, \hat{q}) = \frac{(1-d/b)(1-\hat{q}/f)}{(1-a/b)(1-\hat{q}^{1-n}/f)}\ {}_3\varphi_2(\hat{q}^{-n}, a, b\hat{q}^{-1};\ d, f\hat{q}^{-1};\ \hat{q}, \hat{q})$$
$$+ \frac{(1-d/a)(1-\hat{q}/f)}{(1-b/a)(1-\hat{q}^{1-n}/f)}\ {}_3\varphi_2(\hat{q}^{-n}, a\hat{q}^{-1}, b;\ d, f\hat{q}^{-1};\ \hat{q}, \hat{q}). \qquad (27)$$

This formula is proved in the following way. By comparing coefficients at z^n the following relation is proved:

$$(\alpha - A)\ {}_3\varphi_2(\alpha, A, B;\ d, E;\ \hat{q}, z) = \alpha(1-A)\ {}_3\varphi_2(\alpha, A\hat{q}, B;\ d, E;\ \hat{q}, z)$$
$$-A(1-\alpha)\ {}_3\varphi_2(\alpha\hat{q}, A, B;\ d, E;\ \hat{q}, z).$$

Then we set $z = \hat{q}$, $B = \hat{q}^{-n}$ and apply to every ${}_3\varphi_2$ the symmetry relation (3.2.3) from [15]. Using the renotation $\alpha = d/a$, $A = d/b$ and $E = df/ab$, we obtain relation (27).

Using relation (27) at $\hat{q} = q^{-2}$, $f = q^{2m_1}$, $a = q^{x+m_1}$ and $B = -q^{m_1 - x}$ we derive

$$c^x_m(m_1) = \left(\frac{[x]}{[2x][m_1 - m + 1]}\right)^{1/2} \left\{ q^{-(x+m)/2} \left(\frac{[m_1 - x + 2][x-1]}{[2x-2]}\right)^{1/2} c^{x-1}_m(m_1 + 1)\right.$$
$$\left. + q^{(x-m)/2} \left(\frac{[m_1 + x + 2][x+1]}{[2x+2]}\right)^{1/2} c^{x+1}_m(m_1 + 1)\right\}. \qquad (28)$$

We substitute expression (26) for $c^x_m(m_1)$ into the first sum in (24) and expression (28) into the second sum in (24), and then use formulae (14) and (23) with $m_2 = 0$ and with m_1 replaced by $m_1 - 1$ for the first sum and by $m_1 + 1$ for the second sum. As a result, after some calculation we derive

$$T_{l,\mathbf{0}}(I_{32})|m_1; x\rangle = -A(x)B(x, m_1)|m_1 + 1; x + 1\rangle - A(-x)B(-x, m_1)|m_1 + 1; x - 1\rangle$$
$$+A(x)B(-x-1, m_1 - 1)|m_1 - 1; x + 1\rangle$$
$$+A(-x)B(x-1, m_1 - 1)|m_1 - 1; x - 1\rangle \qquad (29)$$

where $|m_1;x\rangle = q^{-m_1(m_1-l)/2}|m_1;x\rangle''$ and

$$A(x)=\left(\frac{[x][x+1]}{[2x][2x+2]}\right)^{1/2}\qquad B(x,m_1)=q^{(l-2m_1+x-1)/2}([l-m_1][m_1+x+2])^{1/2}.$$

This formula will be used below.

7. The $U_q'(so_3)$ bases of $U_q(sl_3)$ representation spaces

In this section we find the basis of the vector space $V_{l,\mathbf{0}}$ of the representation $T_{l,\mathbf{0}}$ of the algebra $U_q(sl_3)$ for which the operators $T_{l,\mathbf{0}}(I_{21})$ and $T_{l,\mathbf{0}}(I_{32})$ are of the form (6) and (7), respectively. Let

$$|m_1;\ x\rangle = \sum_r b_r^{m_1}(x)|r,x\rangle' \tag{30}$$

where $|m_1;\ x\rangle$ are the basis elements from (29) and $|r,x\rangle'$ are basis elements which must be found, that is,

$$T_{l,\mathbf{0}}(I_{32})|r,x\rangle' = \frac{[x]}{[2x]}([r-x][r+x+1])^{1/2}|r,x+1\rangle'$$
$$-\frac{[x]}{[2x]}([r-x+1][r+x])^{1/2}|r,x-1\rangle'$$

and $T_{l,\mathbf{0}}(I_{21})|r,x\rangle' = \mathrm{i}[x]|r,x\rangle'$. Then

$$T_{l,\mathbf{0}}(I_{32})|m_1;\ x\rangle = \sum_r b_r^{m_1}(x)T_{l,\mathbf{0}}(I_{32})|r,x\rangle'. \tag{31}$$

Substituting here the expression (29) for $T_{l,\mathbf{0}}(I_{32})|m_1;\ x\rangle$ and comparing coefficients at the vector $|r,x+1\rangle'$ and then at the vector $|r,x-1\rangle'$ we obtain the relations

$$C_{x+1}^{m_1-1}b_r^{m_1-1}(x+1) - A_{x+1}^{m_1+1}b_r^{m_1+1}(x+1) = b_r^{m_1}(x)\frac{[x]}{[2x]}([r-x][r+x+1])^{1/2} \tag{32}$$

$$D_{x-1}^{m_1-1}b_r^{m_1-1}(x-1) - B_{x-1}^{m_1+1}b_r^{m_1+1}(x-1) = -b_r^{m_1}(x)\frac{[x]}{[2x]}([r-x+1][r+x])^{1/2} \tag{33}$$

where

$$A_{x+1}^{m_1+1} = A(x)B(x,m_1)\qquad B_{x-1}^{m_1+1} = A(-x)B(-x,m_1)$$
$$C_{x+1}^{m_1-1} = A(x)B(-x-1,m_1-1)\qquad D_{x-1}^{m_1-1} = A(-x)B(x-1,m_1-1).$$

We have from (33) that

$$b_r^{m_1}(x) = \frac{[2x]}{[x]}([r-x+1][r+x])^{-1/2}(B_{x-1}^{m_1+1}b_r^{m_1+1}(x-1) - D_{x-1}^{m_1-1}b_r^{m_1-1}(x-1)).$$

Substituting this expression with (m_1,x) replaced by $(m_1-1,x+1)$ and then by $(m_1+1,x+1)$ into (32), we obtain the recurrence relation for $b_r^{m_1}(x)$ with fixed r and x:

$$A_{x+1}^{m_1+1}B_x^{m_1+2}b_r^{m_1+2}(x) - (A_{x+1}^{m_1+1}D_x^{m_1} + C_{x+1}^{m_1-1}B_x^{m_1})b_r^{m_1}(x)$$
$$+C_{x+1}^{m_1-1}D_x^{m_1-2}b_r^{m_1-2}(x) = -\frac{[x][x+1]}{[2x][2x+2]}[r-x][r+x+1]b_r^{m_1}(x).$$

Introducing the notation

$$N=(l-x)/2\qquad n=(m_1-x)/2\qquad b_r^{m_1}(x)\equiv b_r^{2n+x}(x) = MP_n^x(r)$$

where

$$M = q^{n(x+3/2)} \left(\frac{[2n+2x]!![2N-2n-1]!![2N]!!}{[2n]!![2N-2n]!![2x]!![2N-1]!!} \right)^{1/2}$$

and using the explicit expressions for the coefficients A, B, C and D after some calculation we derive

$$A_n P^x_{n+1}(r) + C_n P^x_{n-1}(r) - (A_n + C_n) P^x_n(r) = -(1 - q^{2(r-x)})(1 - q^{-2(r+x+1)}) P^x_n(r) \qquad (34)$$

where

$$A_n = (1 - q^{-4(n-N)})(1 - q^{-4(n+x+1)})$$
$$C_n = q^{-2(2x+1)}(1 - q^{-4n})(1 - q^{-4(n-N-1/2)}).$$

We compare this relation with the recurrence relation

$$A_n R_{n+1}(\mu(y)) - (A_n + C_n) R_n(\mu(y)) + C_n R_{n-1}(\mu(y)) = -(1 - \hat{q}^{-y})(1 - \gamma\delta\hat{q}^{y+1}) R_n(\mu(y)) \qquad (35)$$

with

$$A_n = (1 - \hat{q}^{n-N})(1 - \gamma\hat{q}^{n+1}) \qquad C_n = \gamma\hat{q}(1 - \hat{q}^n)(\delta - \hat{q}^{n-N-1})$$

for dual q-Hahn polynomials

$$R_n(\mu(y)); \gamma, \delta, N \mid \hat{q}) = {}_3\varphi_2(\hat{q}^{-n}, \hat{q}^{-y}, \gamma\delta\hat{q}^{y+1};\ \gamma\hat{q}, \hat{q}^{-N};\ \hat{q}, \hat{q})$$

where $\mu(y) = \hat{q}^{-y} + \gamma\delta\hat{q}^{y+1}$ (see [22]).

The relation (34) coincides with the recurrence relation (35) if we set

$$\gamma = q^{-4x} \qquad \delta = q^2 \qquad y = (r-x)/2$$

and consider that l, x and r are of the same evenness. (Note that m_1 and x are of the same evenness as shown in section 5.) Therefore,

$$P^x_n(r) = R_n(\mu(y);\ q^{-4x}, q^2, N \mid q^{-4})$$
$$= {}_3\varphi_2(q^{4n}, q^{2r-2x}, q^{-2(x+r+1)};\ q^{-4x-4}, q^{4N};\ q^{-4}, q^{-4}).$$

Since $M P^x_n(r) = b^{m_1}_r(x)$, then for coefficients $b^{m_1}_r(x)$ from (30) we have

$$b^{m_1}_r(x) = q^{n(x+3/2)} \left(\frac{[2n+2x]!![2N-2n-1]!![2N]!!}{[2n]!![2N-2n]!![2x]!![2N-1]!!} \right)^{1/2} R_n(\mu(y);\ q^{-4x}, q^2, N \mid q^{-4}) \qquad (36)$$

where $N = \frac{1}{2}(l-x)$, $n = \frac{1}{2}(m_1 - x)$ and $\mu(y) = q^{2(r-x)} + q^{-2(x+r+1)}$.

The orthogonality relation for the coefficients $b^{m_1}_r(x)$ follows from the orthogonality for the dual q-Hahn polynomials which can be found in [22]. From the last orthogonality we derive that

$$\sum_r b^{m_1}_r(x) b^{m'_1}_r(x) W'(r) = \delta_{m_1 m'_1} \qquad (37)$$

where summation is over $r = x, x+2, x+4, \ldots, l$ and

$$W'(r) = \frac{q^{\frac{1}{2}(r-x)(r+x+1)-(l-x)(x+1)}[l-x]![r+x]![2r+1]}{[r-x]![l+r+1]!![l-r]!![2x]!!}.$$

Formula (37) means that the vectors $|r, x\rangle'$ from (30) are not normalized. The vectors

$$|r, x\rangle = W'(r)^{-1/2}|r, x\rangle'$$

are normalized. This normalization does not change the formula for the operator $T_{l,\mathbf{0}}(I_{32})$. We have

$$|r, x\rangle = \sum_{m_1} b_r^{m_1}(x)W(r)^{1/2}|m_1; x\rangle \qquad |m_1; x\rangle = \sum_{r} b_r^{m_1}(x)W(r)^{1/2}|r, x\rangle.$$

In formula (36), m_1 and r are of the same evenness. If m_1 and r are of different evenness, then we can calculate $b_r^{m_1}(x)$ by means of the formula

$$b_r^{m_1}(x) = q^{(l-2m_1)/2}\left(\frac{[2x][x-1]q^{-x}}{[x][2x-2][r+x][r-x+1]}\right)^{1/2}$$
$$\times\left(q^{-1/2}\sqrt{[l-m_1][m_1-x+2]}\, b_r^{m_1+1}(x-1)\right.$$
$$\left.-q^x\sqrt{[m_1+x][l-m_1+1]}b_r^{m_1-1}(x-1)\right)$$

or by the formula

$$b_r^{m_1}(x) = -q^{(x-l)/2}\left(\frac{[r-x+1][x-1][r+x][m_1+x]}{[x][2x-2][2x][l-m_1+1]}\right)^{1/2} b_r^{m_1-1}(x-1)$$
$$-q^{-(l+x)/2}\left(\frac{[x+1][r-x][r+x+1][m_1-x]}{[2x+2][2x][x][l-m_1+1]}\right) b_r^{m_1-1}(x+1).$$

These formulae are derived from (32) and (33). They express the coefficients $b_r^{m_1}(x)$ with different evenness of m_1 and r in terms of the coefficients with the same evenness of m_1 and r.

8. Representations of $U_q(sl_3)$ in the $U'_q(so_3)$ basis

Now we wish to obtain formulae showing how operators of the representation $T_{l,\mathbf{0}}$ of the algebra $U_q(sl_3)$ act upon the basis $|r, x\rangle$ derived in the previous section. It is enough to derive such a formula only for the operator $T_{l,\mathbf{0}}(q^{2H_3})$ corresponding to the element $q^{2H_3} \in U_q(gl_3)$ (see section 2). The formulae for other operators can be obtained by commuting the operator $T_{l,\mathbf{0}}(q^{2H_3})$ with the operators corresponding to elements of the subalgebra $U'_q(so_3)$. Since $T_{l,\mathbf{0}}(q^{H_3})|m_1;\ m\rangle = q^{m_1-l}|m_1;\ m\rangle$ (see section 4), then

$$T_{l,\mathbf{0}}(q^{2H_3})|m_1;\ x\rangle = q^{2m_1-2l}|m_1;\ x\rangle.$$

We have

$$T_{l,\mathbf{0}}(q^{2H_3})|r, x\rangle = T_{l,\mathbf{0}}(q^{2H_3})\sum_{m_1} W(r)^{1/2}b_r^{m_1}(x)|m_1;\ x\rangle$$
$$= \sum_{m_1} q^{2m_1-2l}\hat{b}_r^{m_1}(x)|m_1;\ x\rangle \tag{38}$$

where $\hat{b}_r^{m_1}(x) = W(r)^{1/2}b_r^{m_1}(x)$. Using the difference equation

$$(\hat{q}^{-n}-1)R_n(\mu(y)) = B(y)R_n(\mu(y+1)) - (B(y)+D(y))R_n(\mu(y)) + D(y)R_n(\mu(y-1))$$

for the dual q-Hahn polynomials $R_n(\mu(y);\ \gamma, \delta, N\,|\,\hat{q})$, where

$$B(y) = \frac{(1-\hat{q}^{y-N})(1-\gamma\hat{q}^{y+1})(1-\gamma\delta\hat{q}^{y+1})}{(1-\gamma\delta\hat{q}^{2y+1})(1-\gamma\delta\hat{q}^{2y+2})}$$

$$D(y) = -\frac{\gamma\hat{q}^{y-N}(1-\hat{q}^{y})(1-\gamma\delta\hat{q}^{y+N+1})(1-\delta\hat{q}^{y})}{(1-\gamma\delta\hat{q}^{2y})(1-\gamma\delta\hat{q}^{2y+1})}$$

(see [22]), we derive from the expression for $\hat{b}_r^{m_1}(x)$ in terms of these polynomials the following recurrence relation for $\hat{b}_r^{m_1}(x)$:

$$q^{-2(l-m_1)}\hat{b}_r^{m_1}(x) = -\hat{B}(r)\hat{b}_{r+2}^{m_1}(x) - \hat{B}(r-2)\hat{b}_{r-2}^{m_1}(x) + \hat{C}(r)\hat{b}_r^{m_1}(x) \tag{39}$$

where

$$\hat{B}(r) = \frac{q-q^{-1}}{q^{(2l+1)/2}}\left(\frac{[l-r][l+r+3][r-x+1][r-x+2][r+x+1]}{[2r+1][2r+5][2r+3]^2[r+x+2]^{-1}}\right)^{1/2}$$

$$\hat{C}(r) = \frac{q^{-l}(q-q^{-1})}{[2r+3]}\left(\frac{q^{r+1}[l-r][r+x+1]}{[2r+1][r+x+2]^{-1}} + \frac{q^{-r}[r-x][r+l+1]}{[2r-1][r-x-1]^{-1}}\right) + q^{2x-2l}.$$

We substitute the expression (39) for $\hat{b}_r^{m_1}(x)$ into (38) and find on the right-hand side the expressions for the vectors $|r+2, x\rangle$, $|r-2, x\rangle$ and $|r, x\rangle$ in terms of the vectors $|m_1;\ x\rangle$. This turns (38) into the formula

$$T_{l,\mathbf{0}}(q^{2H_3})|r, x\rangle = -\hat{B}(r)|r+2, x\rangle - \hat{B}(r-2)|r-2, x\rangle + \hat{C}(r)|r, x\rangle \tag{40}$$

where $\hat{B}(r)$ and $\hat{C}(r)$ are such as in (39). This is the desired formula for the operator $T_{l,\mathbf{0}}(q^{2H_3})$.

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